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PROFESSIONAL PAPER 253 / June 1979

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The d-Choice Secretary Problem

By Kenneth S. Glasser

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"THE d-CHOICE SECRETARY PROBLEM".

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SUMMARY

In the classical Secretary Problem, the player tries to choose the best object of a sequentially ordered set of size N . The value of each object is given by its rank only. At any stage, the player knows the rank of the current object relative to those already seen. Once rejected, an object cannot be chosen later.

In this paper, a generalized Secretary Problem is discussed. The player is given d choices to choose all of the d best objects. The optimal procedure is found by converting the d choice Secretary Problem into a "walk" in a two-dimensional grid. A simple approximation to the optimal strategy rule is also presented.

AMS 1970 subject classifications. 60G40

Key words and phrases. Secretary Problem, Optimal stopping rules, relative ranks.

1. INTRODUCTION

The classical Secretary Problem begins with the premise that N objects or individuals can be ranked from best (rank = 1) to worst (rank = N). As each object is shown to the experimentor (or player), he is able to rank it only in relation to those objects that he has already seen. The player may select the current object, at which point the procedure ends, or he may elect to reject the current object and sample the next. The player also operates under the restriction that once an object has been rejected, it may not be chosen later. If no choice has been made, he is required to choose the N^{th} object.

The optimal procedure for the player to follow depends upon the type of reward he receives. Oftentimes, the player tries to pick the best object. In this case, the optimal procedure maximizes the probability of success. This problem has been solved in a variety of ways, by Lindley (1961), Dynkin (1963), Gilbert and Mosteller (1966), Dynkin and Yushkevich (1969), DeGroot (1970), Chow, Robbins, and Siegmund (1971), and Sirjaev (1973). The problem also appeared in Marvin Gardeners' Scientific American column (Fox and Marnie (1960), Moser and Pounder (1960)),and in the American Mathematical Monthly "Problems and Solutions" (Bissinger and Siegel (1963), Bosch (1964)).

Several authors have discussed the Secretary Problem when the player is given one choice to choose any of the d best objects.

Gilbert and Mosteller (1966) discussed the case $d = 2$. Gusein-Zade (1966), Dynkin and Yushkevich (1969) and Rasmussen (1974 and 1975) have generalized the problem further.

Another related problem that is of interest is to allow the player more than one choice to select the best object. This case has been discussed by Gilbert and Mosteller (1966).

The problem that will be discussed in this paper is a combination of the two generalizations given above. The player is given d choices to select all of the d best objects. We will derive the optimal procedure, which in this case maximizes the probability of selecting the d best. The form of this optimal procedure is:

Procedure A: Choose d starting times $t_1, \ldots, t_d$. Do not select any of the first t_1-1 objects. The d choices are then made in one of three ways:

(1) At any time, select an object that is better than at least one of the objects already chosen.

(2) Make the j^{th} choice of the k^{th} object if $k \geq t_j$, and the k^{th} object has rank j or better, where $1 \leq j \leq d$.

(3) If m choices have been made, $0 \leq m < d$, and $d-m$ objects are left, all of the remaining objects must be chosen.

The procedure ends when the d^{th} choice is made.

The probability of success using Procedure A will be denoted by $P_A(CS; d, t_1, \ldots, t_d, N)$, where CS means correct selection. This will be abbreviated by $P_A(CS)$ when there is no ambiguity. The optimal starting times $t_1^*, \ldots, t_d^*$, are chosen to maximize $P_A(CS)$ for a given N. As in Gilbert and Mosteller (1966), an object that is better than all others seen so far will be called a candidate.

In addition, we will consider an approximation to the optimal rule, based on the optimal starting time for the last choice, t_d^*. The approximately optimal rule has the same form as Procedure A, but the starting times are easier to compute. These approximately optimal starting times do not necessarily maximize $P_A(CS)$.

2. THE d CHOICE PROBLEM - OPTIMAL PROCEDURE

In this section, we will show that the optimal procedure for the d choice Secretary Problem has the form of Procedure A, with starting times $t_1^*, \ldots, t_d^*$. If the player is forced to choose the last d-m objects, as in (3) of Procedure A, he may choose an object that is worse than one already rejected. If such an object is chosen, he cannot succeed in choosing the d best. Unless the player makes choices using (1) or (2), he will not succeed. Thus, without affecting $P_A(CS)$, the type (3) choices can be ignored, since they are included only to insure that d choices are made. When computing $P_A(CS)$, it will be assumed that this has been done, and that choices are made using only (1) and (2); Eliminating type (3) choices will make certain aspects of the construction of the optimal procedure clearer.

Nikolaev (1977) has shown that Procedure A is optimal for the case d=2. In addition, both he and Glasser (1978) have derived exact expressions for $P_A(CS; 2, t_1, t_2, N)$. An exact expression for $P_A(CS; 3, t_1, t_2, t_3, N)$ has also been derived by Glasser. The task of deriving an exact expression for $P_A(CS)$, however, is quite tedious even for d=3.

Theorem 2.1 will show that Procedure A is optimal, and give a method to compute the optimal starting times $t_1^*, \ldots, t_d^*$. In the next section, we will give a specific formulation for t_d^*, and use it to construct an approximation to the optimal rule.

To begin with, define $f = N-d$, and consider the two-dimensional grid of points (i,j), where $i = 0,1,\ldots,d$, and $j = 0,1,\ldots,f$. When the player is at the point (i,j), it means that of $(i+j)$ items sampled so far, he has accepted i and rejected j of them. From the point (i,j), he will move to the point $(i+1,j)$ if he accepts the next object, and $(i,j+1)$ if he rejects it.

Any rule for selecting the d best objects can be put into a 1-1 correspondence with a specific "walk" on the grid. As an example, consider Procedure A. If the player is at the point (i,j), and the $(i+j+1)^{st}$ object is ranked i or better, he will accept it and move to $(i+1,j)$. If the $(i+j+1)^{st}$ object has rank $i+1$, and $i+j+1 \geq t_{i+1}$, then the player also accepts the object and moves to $(i+1,j)$. Otherwise, the object is rejected and the player moves to $(i,j+1)$.

Any time the player "walks" off the grid, he loses the game. For instance, at the point $(d+1,3)$, he has chosen too many objects. At the point $(3,f+1)$, there are not enough objects left to make d choices.

It will be seen in Theorem 2.1 that the optimal procedure requires the player to choose an object better than any object already selected, and reject an object worse than any object already selected. Thus, starting from the point $(0,0)$, the player succeeds if and only if he reaches the point (d,f). If at any time the player chooses an object that is not among the d best, the optimal procedure will force him to choose too many objects. If the player rejects an object that is one of the d best, the optimal procedure will not allow him to make d selections.

A utility function can be defined that reflects the player's desire to choose the d best object. This utility is a function of the ranks of the objects chosen, and is given by:

$$U(i_1, i_2, \ldots, i_d) = \begin{cases} 1 & (i_1, i_2, \ldots, i_d) = (1, 2, \ldots, d) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The player is interested in choosing the d best, and is not concerned with the order of their arrival. Therefore, the ranks of the objects chosen can be permuted in any order without affecting the value of the utility function.

Since the player succeeds in choosing the d best if and only if he reaches the point (d,f), the utility function in (2.1) can be related to the grid described earlier. Assign each point on the grid a conditional expected utility $u(i,j)$. The values $u(i,j)$ are conditioned by the player having accepted i out of $i+j$ objects seen so far. It is also assumed that at each step, the player will elect to continue in an optimal manner. The computation of these $u(i,j)$'s is the subject of the next theorem. Corollary 2.1 shows that Procedure A is the optimal procedure for the d choice Secretary Problem.

Theorem 2.1:

$$u(i,j) = \frac{i\cdot u(i+1,j) + j\cdot u(i,j+1) + \max[u(i,j+1),u(i+1,j)]}{i+j+1}, \tag{2.2}$$

where $i = 0,1,\ldots,d$ and $j = 0,1,\ldots,f$. Initial conditions are given by:

$$\begin{aligned} u(d,f) &= 1, \\ u(d+1,j) &= u(i,f+1) = 0 , \end{aligned} \tag{2.3}$$

for $i = 0,1,\ldots,d-1$, and $j = 0,1,\ldots,f-1$.

Proof: The rank of the $(i+j+1)^{st}$ object can be any of $1,2,\ldots,i+j+1$, with equal probability. This object will be better than at least one of the objects already selected if it has relative rank i or better, with probability $i/(i+j+1)$. When the object is better than an object already selected, it must be selected if the player wishes to continue the possibility of selecting the d best.

Otherwise, if the player rejects an object ranked i or better, one of two possibilities can occur. If the $(i+j+1)^{st}$ object is not one of the d best, he has already chosen an object that cannot be one of the d best. If the $(i+j+1)^{st}$ object is one of the d best, the player will have rejected an object that is one of the d best. In either case, the player cannot choose all of the d best.

Thus, an object ranked between 1 and i must be accepted. In this case, the player moves from (i,j) to $(i,j+1)$.

When the $(i+j+1)^{st}$ object is worse than at least one of the objects already selected, with relative rank between $i+2$ and $i+j+1$, it is better to reject it. The reasons are similar to those given above. The probability that the rank is $i+2$ or greater is $j/(i+j+1)$. The player in this case moves from (i,j) to $(i,j+1)$.

A question arises then, only in the case that the $(i+j+1)^{st}$ object is neither better than any objects already selected, nor worse than any already rejected. This means that the rank of the new object is $i+1$, and this occurs with probability $1/(i+j+1)$.

For this last case, the best course of action depends on the expected utility if the object is rejected, $u(i,j+1)$, and the expected utility if the object is selected $u(i+1,j)$. Clearly, if $u(i+1,j) > u(i,j+1)$ it is better to accept the new object. When $u(i+1,j) < u(i,j+1)$ it is better to reject the new object. When the utilities are equal, the choices are equivalent.

So the conditional expected utility at (i,j) is given by (2.2). Using (2.2), with initial conditions (2.3), the values of $u(i,j)$ can be computed backwards beginning with $u(d,f) = 1$.

The optimal procedure, following the reasoning of the above paragraphs, is given by:

<ins>Procedure A'</ins>: Accept the $(i+j+1)^{st}$ object if it is better than at least one of the objects already selected. Reject the

$(i+j+1)^{st}$ object if it is worse than at least one object already rejected. Otherwise, accept or reject the $(i+j+1)^{st}$ object if $u(i+1,j)$ is greater than or less than $u(i,j+1)$ respectively.

As an example, assume that the player is to pick the best two out of seven objects. The grid for the optimal procedure is given below, with $N = 7$, $d = 2$, $f = 5$:

5	$\frac{1}{21}$	$\frac{2}{7}$	1
4	$\frac{13}{105}$	$\frac{9}{21}$	$\frac{5}{7}$
3	$\frac{86}{420}$	$\frac{47}{105}$	$\frac{10}{21}$
2	$\frac{343}{1260}$	$\frac{171}{420}$	$\frac{30}{105}$
1	$\frac{745}{2520}$	$\frac{402}{1260}$	$\frac{60}{420}$
0	$\frac{745}{2520}$	$\frac{462}{2520}$	$\frac{60}{1260}$
	0	1	2

The arrows connect the path with the maximal expected utilities.

It will be shown in the next corollary that Procedure A with starting times $t_1^*,\ldots,t_d^*$ is the same as Procedure A'. The following definition will be helpful. Consider the diagonal set of points $(0,m)$, $(1,m-1)$, $(2,m-2)$, ..., $(m-1,1)$, $(m,0)$. Call the set of points on this diagonal that are also on the grid an axis of constant sample size m. When no ambiguity exists, the term will be abbreviated as an axis of size m, or simply called an axis. In the above example, the points $(0,5)$, $(1,4)$, and $(2,3)$ are an axis of size 5.

Note that in the above example, there is one maximal utility on each axis, and the utilities decrease for points further away from the optimal path. This is the subject of the following corollary.

Corollary 2.1: On each axis of size m, $m = 0,1,\ldots,N-1$, $u(i,j)$ is a unimodal function of i, and is monotonically decreasing on each side of the mode.

Proof: By backward induction. Assume that $d < f$. By (2.2), $u(d-1,f) = d/N < f/N = u(d,f-1)$. Thus, the corollary is true for $m = N-1$. Assume that the corollary is true for $m+1$, and let $i+j = m$. Consider the points $(i-1, j+2)$, $(i,j+1)$, and $(i+1,j)$, and assume that they lie above the optimal path. Then the conditional expected utilities for these points satisfy $u(i-1,j+2) = a < u(i,j+1) = b < u(i+1,j) = c$.

From (2.2), the following may be computed:

$$u(i-1,\ j+1) = \frac{ib + (j+1)a}{i+j+1} ,$$

$$u(i,j) = \frac{(i+1)c + jb}{i+j+1} .$$

Since $a < b < c$, $u(i,j) - u(i-1,\ j+1) > 0$.

In a similar manner, it can be shown that for points below the optimal path, as the first argument increases the expected utility decreases. This completes the proof.

Corollary 2.1 shows that above the optimal path, the player will select an object when given a choice. Below the optimal path, he will reject an object when given a choice. Using the optimal strategy A', the player will attempt to stay as close as possible to the optimal path.

In other words, if the j^{th} choice has not been made, and the k^{th} object is ranked j^{th}, where $k \geq t_j^*$, the optimal strategy A' says to choose the object. Otherwise select objects better than at least one object already selected, and reject any object that is worse than any object already rejected. This is the same as Procedure A with starting times $t_1^*,\ldots,t_d^*$.

In the example computed before, the optimal starting times are given by $t_1^* = 2$, $t_2^* = 5$. The example is also an application of Corollary 2.1.

Using the utility function (2.1), it can be seen that the probability of success given that the player is at the point (d,f) is the same as the utility $u(d,f) = 1$. Working backwards, the conditional expected utility $u(d-1,f)$ is the same as the conditional probability that the player wins, given that he is at the point $(d-1,f)$ and that he will continue to the last step in an optimal manner. Continuing to work backwards, the expected utility $u(0,0)$ is the same as the probability that the player wins using the optimal starting times $t_1^*,\ldots,t_d^*$, and making the optimal choice at each step:

$$u(0,0) = P_A(CS;\ d,t_1^*,\ldots,t_d^*,N)\ .$$

In the above example, this probability is given by $u(0,0) = 745/2520 = .295635$.

Tables 1 and 2 contain the optimal starting times and the probability of success for the d choice Secretary Problem for $d = 2,5$ and $N = d(1)\ 50(10)\ 100,\ 1000$. All of these calculations were done using Theorem 2.1. As expected, when $d = 2$, the results are the same as those derived using the exact expression of Nikolaev (1977) or Glasser(1978).

In Tables 1 and 2, it appears that the optimal probability of success is a non-increasing function of N. This is the subject of the next theorem and corollary.

Theorem 2.2: If $d < N$,

$$P_A(CS;\ d,t_1,\ldots,t_d,N) \le \frac{1}{N}\sum_{i=1}^{N} P_A(CS;\ d,t_1^{(i)},\ldots,t_d^{(i)},N-1) ,$$

where
$$t_j^{(i)} = t_j \qquad \text{if } i \ge t_j$$
$$= t_j - 1 \qquad \text{if } i < t_j .$$

Proof: Among N applicants, the probability that the worst is in position i is $1/N$, so that $P_A(CS;\ d,t_1,\ldots,t_d,N) = \sum_{i=1}^{N} P_A(CS;\ d,t_1,\ldots,t_d,N|i)$, where $P_A(CS;\ d,t_1,\ldots,t_d,N|i)$ is the conditional probabiltiy of success for procedure A given that the worst (rank = N) object is sampled i^{th}.

Consider two cases:

Case (a): $t_1 > 1$ (so the first object is not chosen). For any value of i, the worst object will not be chosen, since it is worse than an object already rejected (the first). Thus, in this case, the i^{th} object could be ignored without affecting the probability of success, provided we renumber properly, i.e., $P_A(CS;\ d,t_1,\ldots,t_d,N|i) = P_A(CS;\ d,t_1^{(i)},\ldots,t_d^{(i)},\ N-1)$.

Case (b): $t_1 = 1$(so the first object is chosen). The same argument shows that $P_A(CS;\ d,t_1,\ldots,t_d,N|i) = P_A(CS;\ d,t_1^{(i)},\ldots,t_d^{(i)},N-1)$, provided i is not too small, namely if $i>d$ or $t_i>i$. Otherwise, $i \le d$ and $t_j = j$ for all $j \le i$, in which case all of the first i objects are chosen, including the worst, so

$$P_A(CS;\ d,t_1,\ldots,t_d,N) = 0 \le P_A(CS;\ d,t_1^{(i)},\ldots,t_d^{(i)},N-1) .$$

Corollary 2.2: $P_A(CS;\ d,t_1^*,\ldots,t_d^*,N) \le P_A(CS;\ d,t_1^*,\ldots,t_d^*,N-1)$.

Proof: The corollary follows directly from Theorem 2.2.

We have seen that $P_A(CS;\ d,t_1^*,\ldots,t_d^*,d+1) \ge P_A(CS;t_1^*,\ldots,t_d^*,d+2) \ge P_A(CS;\ t_1^*,\ldots,t_d^*,N) \ge \ldots$ We shall now compute a lower bound for the probability of success. Consider the strategy which rejects the first r and picks when possible anything after that. One reaches $(0,r)$ with probability 1 and beyond that the probabilities at (i,j) are $(i+1)/(i+j+1)$ to $(i+1,j)$ and $j/(i+j+1)$ to $(i,j+1)$. Any path from $(0,r)$ to $(d,N-d)$ will have the combined probability $1 \cdot 2 \cdots d \cdot r(r+1) \cdots (N-d-1)/((r+1)(r+2) \cdots N)$ and there are $\binom{N-r}{d}$ such paths, so the total probability of success by this strategy is

$$\frac{(N-r) \cdots (N-r-d+1)}{d!} \frac{d!r}{N \cdots (N-d)} = \frac{r}{N-d} \left(1-\frac{r}{N}\right) \cdots \left(1-\frac{r}{N-d+1}\right).$$

If we choose $r = \left[(N-d)/(d+1)\right]$ where $[X]$ is the greatest integer less than X, then the product converges to:

$$\frac{1}{d+1} \left(1-\frac{1}{d+1}\right)^d > \frac{1}{e(d+1)} .$$

There is an obvious symmetry between the optimal strategies for choosing the best d out of N and the best $N-d$ out of N. In particular, for $N=2d$, we have $u(i,j) = u(j,i)$ and the maximum on each diagonal $i+j = r$ occurs at $(r/2, r/2)$ for even r and $((r+1)/2, (r-1)/2)$ and $((r-1)/2, (r+1)/2)$ for odd r. In other words, choose, if possible, when $i < j$, reject if possible, when $i > j$. When $i = j$, either decision will do.

Knowing the optimal strategy for the case $N=2d$, the probability of success can be calculated by starting at $(0,0)$. Let $p(i,j)$ be the probability of getting from $(0,0)$ to (i,j). Since one can only get to (i,j) from either $(i-1,j)$ or $(i,j-1)$, we have:

$$p(i,j) = p_c(i-1,j)\, p(i-1,j) + p_r(i,j-1)\, p(i,j-1) , \tag{2.4}$$

where $p_c(i,j)$ is the probability of choosing an object when at (i,j) and $p_r(i,j)$ is the probability of rejecting an object when at (i,j).

$$
\begin{aligned}
p_c(i,j) &= (i+1)/(i+j+1) && \text{if } i < j \\
&= i/(i+j+1) && i > j \\
&= \tfrac{1}{2} && i = j.
\end{aligned}
$$

$$
\begin{aligned}
p_r(i,j) &= j(i+j+1) && \text{if } i < j \\
&= (j+1)/(i+j+1) && i > j \\
&= \tfrac{1}{2} && i = j.
\end{aligned}
$$

Then, by induction beginning at $(0,0)$, it can be shown that:

$$
\begin{aligned}
p(i,j) &= \frac{i+j+1}{2i(i+1)} && \text{for } i > j \\
&= \frac{i+j+1}{2j(j+1)} && i < j \\
&= \frac{1}{i+1} && i = j.
\end{aligned}
$$

(Do the cases $i=j$, $i=j+1$, $i>j+1$ separately; the rest follow by symmetry.) In particular, $p(d,d) = 1/(d+1) = P_A(CS;\ d,t_1^*,\ldots,t_d^*,2d)$.

The case $N = 2d+1$ can be handled similarly once the strategy is determined. Using an inductive proof similar to that of corollary 2.1, it can be shown that the maximum occurs at

$(r/2, r/2)$ or $((r-1)/2, (r+1)/2)$ depending on whether r is even or odd. (More specifically, it can be shown that for even r, $u(r/2,r/2) > u((r-2)/2,(r+2)/2) > u((r+2)/2,(r-2)/2) > \ldots > u((r-2i)/2,(r+2i)/2) > u((r+2i)/2,(r-2i)/2)$ and for odd r, $u((r-1)/2,(r+1)/2) > u((r+1)/2,(r-1)/2) > \ldots > u((r-2i-1)/2, (r-2i+1)/2) > u((r-2i+1)/2, (r-2i-1)/2) > \ldots)$.

The strategy, therefore, is to choose, if possible when $i < j$, and reject if possible when $i \geq j$. Knowing this, we may proceed as in the case $N = 2d$. For this case, the values of $p_c(i,j)$ and $p_r(i,j)$ are given by:

$$\begin{aligned} p_c(i,j) &= (i+1)/(i+j+1) && \text{if } i < j \\ &= i/(i+j+1) && i \geq j. \end{aligned}$$

$$\begin{aligned} p_r(i,j) &= j/(i+j+1) && \text{if } i < j \\ &= (j+1)/(i+j+1) && i \geq j. \end{aligned}$$

Starting with $(0,0)$ and using (2.4), we can show by induction that

$$\begin{aligned} p(i,j) &= 1/j && \text{if } i < j \\ &= j/i(i+1) && i \geq j. \end{aligned}$$

(Do the cases $i < j-1$, $i = j-1$, $i = j$ and $i > j$ separately.)

Thus, $P_A(CS;\ d,t_1^*,\ldots,t_d^*,2d+1) = 1/(d+1)$. Note the coincidence that $P_A(CS;\ d,t_1^*,\ldots,t_d^*,2d) = P_A(CS;\ t_1^*,\ldots,t_d^*,2d+1)$.

3. RESULTS FOR THE LAST CHOICE

When the formulation of the previous section is used, it is possible to determine a general formula for the d^{th} starting time. It is easier to do this if some new notation is introduced. Let:

$$b(i,j) = \frac{u(i,j)}{i} \frac{\binom{N}{d}}{\binom{i+j}{j}} , \tag{3.1}$$

$$c(i,j) = \frac{u(i,j)}{j} \frac{\binom{N}{d}}{\binom{i+j}{j}} .$$

By the definitions of $b(i,j)$ and $c(i,j)$,

$$c(i,j) = \frac{i}{j} b(i,j) .$$

The following result shows why this new notation is useful.

Lemma 3.1: (1) If $b(i+1,j) < c(i,j+1)$, then

$$b(i,j) = b(i+1,j) + b(i,j+1) . \tag{3.2}$$

(2) If $b(i+1,j) > c(i,j+1)$, then,

$$c(i,j) = c(i+1,j) + c(i,j+1) . \tag{3.3}$$

Proof: First, to show both (1) and (2), it is necessary to show that $b(i+1,j) <$ or $> c(i,j+1)$ if and only if $u(i+1,j) <$ or $> u(i,j+1)$. By (3.1), these assertions must be true, since:

$$b(i+1,j) - c(i,j+1) = \frac{\binom{N}{d}}{\binom{i+j+1}{j}(i+1)} \Big[u(i+1,j) - u(i,j+1)\Big] .$$

To show (1), if $b(i+1,j) < c(i,j+1)$, then $u(i+1,j) < u(i,j+1)$. From (3.2), this means that:

$$u(i,j) = \frac{i}{i+j+1} u(i+1,j) + \frac{j+1}{i+j+1} u(i,j+1) . \tag{3.4}$$

Multiplying both sides of (3.2) by $\binom{N}{d}/i\binom{i+j}{j}$ gives the result. (2) is proved in a similar manner.

Lemma 3.1 says that below the optimal path, the $b(i,j)$'s can be found using (3.2), while on or above the optimal path the $c(i,j)$'s can be found using (3.3). Using Lemma 3.1, an expression for the last starting time is derived in the next theorem.

Theorem 3.1: The last optimal starting time is given by:

$$t_d^* = \min\left\{m \,\middle|\, \sum_{k=m+1}^{N} \frac{1}{k-d} < \frac{1}{d}\right\} . \tag{3.5}$$

<u>Proof</u>: Assume that $f > d$. From (2.2), it can be seen that:

$$u(i,f) = \binom{i+f}{i} / \binom{N}{d},$$

for $i = 0,1,\ldots,d-1$. Thus, from (3.1),

$$c(i,f) = \frac{1}{f} \qquad i = 0,1,\ldots,d-1.$$

Again using (2.2), the conditional expected utilities of the last column are given by:

$$u(d,j) = \binom{d+j}{j} / \binom{N}{d}, \tag{3.6}$$

for $j = 0,1,\ldots,f-1$. So from (3.1), the last column of $c(i,j)$'s is given by:

$$c(d,j) = 1/j \qquad j = 1,2,\ldots,f-1.$$

Using (3.1) and (3.6), it can be seen that:

$$b(d,j) = 1/d \qquad j = 0,1,\ldots,f-1.$$

Now consider the values of $c(i,j)$ and $b(i,j)$ for the next to last column. Start with the point $(d-1,f-1)$ and decrease the second argument one step at a time. As long as $c(d-1,j) <$ $b(d,j-1) = 1/d$ Lemma 3.1 gives:

$$c(d-a,j) = \sum_{k=j}^{f} \frac{1}{k} .$$

The partially computed grid of $c(i,j)$'s looks like:

$$
\begin{array}{lccccc}
f & \frac{1}{f} & \frac{1}{f} & \cdots & \frac{1}{f} & \frac{1}{f} \\[2ex]
f-1 & & & & \frac{1}{f} + \frac{1}{f-1} & \frac{1}{f-1} \\[2ex]
f-2 & & & & \frac{1}{f} + \frac{1}{f-1} + \frac{1}{f-2} & \frac{1}{f-2} \\[2ex]
\vdots & & & & \vdots & \vdots \\[2ex]
j & & & & \sum_{k=j}^{f} \frac{1}{k} & \frac{1}{j} \\[2ex]
 & 0 & 1 & \cdots & d-1 & d
\end{array}
$$

The last choice begins to be desirable at q, where q is the first row that:

$$c(d-1,q) = \sum_{k=q}^{f} \frac{1}{k} > \frac{1}{d} = b(d,q-1) \ . \tag{3.7}$$

At the point $(d-1,q)$, $d+q-1$ objects have been sampled. So the last starting time is given by $d+q$. The sum in (3.7) can be rewritten as $\sum_{k=q+d}^{N} \frac{1}{k-d}$, since by definition, $f = N-d$. By defining $m = q+d$, the definition of t^* can be written as:

$$t_d^* = \max\left\{q \,\middle|\, \sum_{k=q}^{f} \frac{1}{k} > \frac{1}{d}\right\} + d \ . \tag{3.8}$$

Equation (3.8) is equivalent to (3.5).

Theorem 3.1 is a generalization of results given in Gilbert and Mosteller (1966) and Lindley (1961) for $d=1$, and the results found in Glasser (1978) for $d=2$. Bounds can be placed on t_d^*, and are given in the next corollary.

Corollary 3.1:

$$(N-d+1)\, e^{-1/d} + d-1 < t_d^* < (N-d)\, e^{-1/d} + d+1 . \tag{3.9}$$

Proof: From (3.5), for $m=t_d^*$,

$$\sum_{k=m}^{N} \frac{1}{k-d} > \frac{1}{d} . \tag{3.10}$$

$$\sum_{k=m+1}^{N} \frac{1}{k-d} < \frac{1}{d} . \tag{3.11}$$

For any decreasing integrable function $h(x)$, defined on the real numbers, we have:

$$\int_a^{b+1} h(x)dx < \sum_{k=a}^{b} h(k) < \int_{a-1}^{b} h(x)dx. \tag{3.12}$$

Applying (3.12) to (3.10) and (3.11) gives:

$$\frac{1}{d} < \log \frac{N-d}{m-d-1} \tag{3.13}$$

$$\frac{1}{d} > \log \frac{N-d+1}{m-d+1} . \tag{3.14}$$

Rearranging (3.13) and (3.14) completes the proof.

It can be seen from Corollary 3.1 that if N increases to infinity while d remains constant, the following is true:

$$\lim_{N\to\infty} t_d^*/N = e^{-1/d} .$$

The importance of this section is shown in the next section. There, a rule will be given to approximate the optimal starting times. This rule will allow the player to calculate approximately optimal starting times much more readily than the true optimal starting times.

4. THE LAST CHOICE RULE

Even for $d=2$, computing the grid in Section 2 is more direct and is a faster way to find the optimal starting times than the method outlined in Nikolaev (1977) or Glasser(1978). Computational difficulties will still arise, however, for the player who does not have access to a computer. It will be seen shortly that the approximation developed in this section is close even for moderately sized N.

The optimal last starting time, t_d^*, is given in Theorem 3.1. From corollary 3.1, it can be seen that t_d^* is contained in an interval of length $2 - e^{-1/d}$, a value between 1 and 2. As d becomes larger, the size of this interval becomes closer to 1. An approximation to t_d^* is given by:

$$t_d' = \left[(N - d+1)\, e^{-1/d} + d\right] , \qquad (4.1)$$

where $[X]$ denotes the greatest integer less than or equal to X.

The value of t_d' differs by at most one from the value of t_d^*. Most of the time, especially as d becomes larger, t_d^* and t_d' will be the same. The advantage to using t_d' is its computational ease, especially when N is at all large. The Last Choice Rule can now be defined.

Procedure B (The Last Choice Rule): d starting times are chosen in the following manner: the d^{th} starting time is found using (4.1). By redefining N as $t_d'-1$, and d as $d-1$, the next-to-last starting time is found by again applying (4.1), or:

$$t_{d-1}' = \left[(t_d' - d+1)\, e^{-1/(d-1)} + d-1\right] .$$

Continue in this manner until all d starting times have been computed. The player then chooses objects on the same basis as Procedure A.

The probabiltiy of success using the Last Choice Rule will be denoted by $P_B(CS)$. The Last Choice Rule is not optimal, unless

$$t_i' = t_i^*, \; i = 1, 2, \ldots, d.$$

Tables 1 and 2 give the starting times and the probability of success for both the Optimal Rule and the Last Choice Rule. Values of d are 2 and 5, and $N = d(1)\, 50(10)\, 100,\ 1000$.

The computation of the probability of success when the Last Choice Rule is used is similar to the method given in Section 2. The conditional utilities of the grid are given by:

$$u(i,j) = \begin{cases} \dfrac{(i+1)\cdot u(i+1,j) + j \cdot u(i,j+1)}{i+j+1} & \text{if } \; i+j+1 \geq t_{i+1} \\[2ex] \dfrac{i \cdot u(i,j+1) + (j+1) \cdot u(i,j+1)}{i+j+1} & \text{if } \; i+j+1 < t_{i+1} . \end{cases} \qquad (4.2)$$

The difference between (4.2) and (2.2) is that on the optimal grid, given a choice, the player chooses the point with the larger expected utility. If the starting times are assigned beforehand, as in the Last Choice Rule, the player when given a choice, is forced to select an object if the starting number has been reached.

Tables 1 and 2 show the accuracy of the approximations. In these tables, the worst percentage decrease using the Last Choice Rule for $N \geq 10$ is 4.2 percent, when $d=5$ and $N=21$. For $d=2$, as long as N is greater than 14 the percentage decrease is well under 1 percent. The percentage decrease is under 1 percent for $d=3$ (not shown) when $N > 20$, for $d=4$ (not shown) when $N > 26$, and for $d=5$ when $N > 32$. For N larger than 100, the decrease is reduced to less than .5 percent in almost all cases.

While the percentage decrease in the probability of success using the Last Choice Rule is an increasing function of d, it appears from the calculations done so far that this error is still very reasonable. For the statistician who does not have access to a computer, it is an easy way to approximate the optimal starting times using only a hand-calculator or tables of the exponential function.

For those with access to a computer, programming algorithm (2.2) is not difficult, and it is a fast algorithm to run. This would, of course, negate the need for the Last Choice Rule as an approximation.

TABLE 1

STARTING TIMES AND PROBABILITY OF SUCCESS FOR THE OPTIMAL AND LAST CHOICE RULES, d=2

	Optimal Rule		Last Choice Rule	
N	Starting Times	$P_A(CS)$	$P_B(CS)$	Starting Times
2	1,2	1.000000	1.000000	1,2
3	1,3	.500000	.500000	1,3
4	1,3	.333333	.333333	2,4
5	2,4	.333333	.333333	2,5
6	2,5	.313889	.313889	2,5
7	2,5	.295635	.295635	2,5
8	2,6	.279960	.279960	2,6
9	3,7	.273898	.262720	2,6
10	3,7	.271433	.271433	3,7
11	3,8	.266299	.266299	3,8
12	3,8	.260400	.260400	3,8
13	4,9	.257516	.254872	3,9
14	4,10	.255854	.248040	3,9
15	4,10	.254142	.254142	4,10
16	4,11	.251513	.251513	4,11
17	5,11	.248729	.248336	4,11
18	5,12	.248519	.248519	5,12
19	5,13	.247338	.247067	5,12
20	5,13	.246057	.246057	5,13
21	5,14	.244357	.244357	5,14
22	6,14	.243660	.242263	5,14
23	6,15	.243309	.243309	6,15
24	6,16	.242398	.242186	6,15
25	6,16	.241379	.241379	6,16
26	7,17	.240668	.240144	6,17
27	7,17	.240293	.238609	6,17
28	7,18	.239916	.239916	7,18
29	7,19	.239176	.239002	7,18
30	8,19	.238389	.238332	7,19
31	8,20	.238299	.237370	7,20
32	8,20	.237901	.236168	7,20
33	8,21	.237532	.237532	8,21
34	8,22	.236909	.236909	8,22
35	9,22	.236661	.236190	8,22
36	9,23	.236509	.236509	9,23
37	9,24	.236133	.236115	9,23
38	9,24	.235766	.235766	9,24
39	10,25	.235410	.235228	9,25
40	10,25	.235292	.234601	9,25

TABLE 1
(Continued)

N	Optimal Rule: Starting Times	Optimal Rule: $P_A(CS)$	Last Choice Rule: $P_B(CS)$	Last Choice Rule: Starting Times
41	10,26	.235111	.235111	10,26
42	10,27	.234762	.234733	10,26
43	10,27	.234405	.234405	10,27
44	11,28	.234343	.233933	10,28
45	11,28	.234184	.233377	10,28
46	11,29	.233991	.233991	11,29
47	11,30	.233666	.233632	11,29
48	12,30	.233539	.233325	11,30
49	12,31	.233452	.233452	12,31
50	12,31	.233271	.233271	12,31
60	14,37	.231853	.231853	14,37
70	17,44	.230943	.230807	16,43
80	19,50	.230274	.230004	18,49
90	21,56	.229714	.229367	20,55
100	24,62	.229260	.229241	23,62
1000	230,608	.225816	.225739	223,607

TABLE 2

STARTING TIMES AND THE PROBABILITY OF SUCCESS FOR THE OPTIMAL AND LAST CHOICE RULES, d=5

	Optimal Rule		Last Choice Rule	
N	Starting Times	$P_A(CS)$	$P_A(CS)$	Starting Times
5	1,2,3,4,5	1.000000	1.000000	1,2,3,4,5
6	1,2,4,5,6	.427778	.380556	1,3,4,5,6
7	1,3,4,6,7	.295635	.275397	1,3,5,6,7
8	1,3,5,6,8	.238194	.231250	1,3,5,7,8
9	1,3,5,7,9	.200000	.200000	1,3,5,7,9
10	2,4,6,8,10	.166667	.166667	1,3,5,7,9
11	2,4,6,8,10	.166667	.166667	2,4,6,8,10
12	2,4,6,9,11	.161551	.156249	2,5,7,9,11
13	2,4,7,9,12	.156208	.151789	2,5,8,10,12
14	2,5,7,10,13	.151445	.149512	2,5,8,11,13
15	2,5,8,11,14	.147286	.147286	2,5,8,11,14
16	2,5,8,11,14	.142944	.142944	2,5,8,11,14
17	2,5,9,12,15	.138755	.138676	2,6,9,12,15
18	2,6,9,13,16	.135380	.134829	2,6,10,13,16
19	3,6,10,13,17	.132204	.131596	2,6,10,14,17
20	3,6,10,14,18	.131297	.128442	2,6,10,14,18
21	3,7,11,15,19	.129967	.124524	2,6,10,14,18
22	3,7,11,15,19	.129375	.129375	3,7,11,15,19
23	3,7,11,16,20	.128149	.127480	3,8,12,16,20
24	3,7,12,16,21	.126831	.126242	3,8,13,17,21
25	3,8,12,17,22	.125684	.125346	3,8,13,18,22
26	3,8,13,18,23	.124446	.124446	3,8,13,18,23
27	3,8,13,18,23	.123047	.123047	3,8,13,18,23
28	3,9,14,19,24	.121769	.121769	3,9,14,19,24
29	4,9,14,20,25	.121528	.120410	3,9,15,20,25
30	4,9,15,20,26	.121159	.119159	3,9,15,21,26
31	4,9,15,21,27	.120655	.117884	3,9,15,21,27
32	4,10,16,22,28	.120110	.116301	3,9,15,21,27
33	4,10,16,22,28	.119700	.119700	4,10,16,22,28
34	4,10,17,23,29	.119072	.118888	4,11,17,23,29
35	4,11,17,23,20	.118428	.118246	4,11,18,25,30
36	4,11,17,24,31	.117841	.117721	4,11,18,24,31
37	4,11,18,25,32	.117188	.117188	4,11,18,25,32
38	5,11,18,25,32	.116839	.116434	4,11,18,25,32
39	5,12,19,26,33	.116713	.116713	5,12,19,26,33
40	5,12,19,27,34	.116508	.116453	5,12,20,27,34
41	5,12,20,27,35	.116233	.116194	5,12,20,28,35
42	5,13,20,28,36	.115891	.115886	5,12,20,28,36
43	5,13,21,29,37	.115570	.115384	5,12,20,28,36

TABLE 2
(Continued)

N	Optimal Rule: Starting Times	$P_A(CS)$	$P_B(CS)$	Last Choice Rule: Starting Times
44	5,13,21,29,37	.115264	.115264	5,13,21,29,37
45	5,13,22,30,38	.114858	.114797	5,14,22,30,38
46	5,14,22,30,39	.114466	.114379	5,14,23,31,39
47	6,14,22,31,40	.114241	.114015	5,14,23,32,40
48	6,14,23,32,41	.114135	.113642	5,14,23,32,41
49	6,14,23,32,41	.113944	.113145	5,14,23,32,41
50	6,15,24,33,42	.113844	.113844	6,15,24,33,42
60	7,18,28,39,50	.112087	.111178	6,17,28,39,50
70	8,20,33,46,59	.110912	.110156	7,20,33,46,59
80	9,23,38,52,67	.110031	.110018	9,23,38,53,67
90	10,26,42,59,75	.109350	.109338	10,26,32,59,75
100	11,29,47,65,83	.108808	.108808	11,29,47,65,83
1000	108,281,460,639,820	.104736	.104612	102,278,458,639,820

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**Research supported in part under Office of Naval Research Contract N00014-68-0273-0017

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